

NPI MSU 98-54/555
hep-ph/9812529
December 1998

The strategy of regions for asymptotic expansion of two-loop vertex Feynman diagrams

V.A. Smirnov¹ and E.R. Rakhmetov²

*Nuclear Physics Institute of Moscow State University
Moscow 119899, Russia*

Abstract

General prescriptions for evaluation of coefficients at arbitrary powers and logarithms in the asymptotic expansion of Feynman diagrams in the Sudakov limit are discussed and illustrated by two-loop examples. Peculiarities connected with evaluation of individual terms of the expansion, in particular, the introduction of auxiliary analytic regularization, are characterized.

¹E-mail: smirnov@theory.npi.msu.su. Supported by the Russian Foundation for Basic Research, project 98-02-16981 and by Volkswagen Foundation, contract No. I/73611.

²E-mail: rahmetov@theory.npi.msu.su. Supported by the Russian Foundation for Basic Research, project 98-02-16981.

1. The simplest explicit formulae [1–3] (see brief reviews in [4]) for the asymptotic expansion of Feynman diagrams in various off-shell limits of momenta and masses, when the momenta are considered either large or small in the Euclidean sense, have been generalized to some typically Minkowskian on-shell limits [5–7], in particular, to the Sudakov limit. The prescriptions for these limits have been formulated using (pre)subtractions in a certain family of subgraphs of a given graph.

Recently explicit prescriptions for expanding Feynman integrals near threshold have been presented [8]. This was done with the help of a standard physical strategy based on analysis of regions in the space of loop momenta. It should be pointed out, however, that this strategy of regions is usually applied only for evaluating and summing up the leading logarithms, in particular, in the Sudakov limit (see. e.g. [9]). Note that the information about the leading logarithms is present only in contributions of some specific regions so that usually one does not consider integration in other domains.

It was argued (and demonstrated for the threshold expansion) in [8] that it is worthwhile to use this strategy for the evaluation of coefficients at any power and logarithm for an arbitrary limit. In such extended form, the strategy reduces to the following prescriptions:

- (a) consider all the regions of the loop momenta that are typical for the given limit and expand, in every region, the integrand in a Taylor series with respect to the parameters that are considered small in the given region,
- (b) integrate the integrand expanded, in every region in its own way, over the whole integration domain in the loop momenta,
- (c) put to zero any scaleless integral (even if it is not regulated, e.g., within dimensional regularization).

As it was pointed out in [8], it is the step (b) in this procedure that is far from being trivial. One may believe that this strategy is legitimate for every limit of momenta and masses. For example, it leads to the well-known formulae for asymptotic expansions in the case of typically Euclidean limits [1, 2] (proven in [3]) so that we have such indirect confirmation at least for them. Note that, for these limits as well as for the on-shell limit considered in refs. [5, 6], the collection of relevant regions is determined by subdividing all the loop momenta into large (hard) and small (soft) ones.

In the present paper, we check, by two-loop examples, this heuristic procedure for the evaluation of coefficients at arbitrary powers and logarithms in asymptotic expansions of Feynman diagrams in the Sudakov limit [10]. We shall consider two commonly accepted variants of this limit for vertex diagrams with the external momenta p_1, p_2 and $q = p_1 - p_2$:

Limit 1. Two external momenta are off shell, $p_1^2 = p_2^2 = m^2 = -\mu^2$, $Q^2 \equiv -q^2 \rightarrow \infty$, all internal masses are zero.

Limit 2. Two external momenta are on shell, $p_1^2 = p_2^2 = 0$; $Q^2 \rightarrow \infty$; some internal masses are non-zero.

We shall calculate the leading power behaviour, including all the logarithms,

$\ln^j(q^2/m^2)$, $j = 0, 1, 2, 3, 4$, of the massless planar diagram Fig. 1a in the first limit and compare the obtained result with a known explicit expression [11]. After this confirmation we shall then apply the above heuristic prescription to the non-planar diagram, Fig. 1b, (for which no analytical results are known) in Limit 2 when $m_1 = \dots = m_4 = 0$, $m_5 = m_6 = m$. We shall also use the second example to describe techniques for evaluation of individual terms of the expansion. A natural way for evaluation terms with the $1/(m^2)^{2\epsilon}$ dependence is introduction of an auxiliary analytical regularization. In contrast to the planar diagram in the second limit where the poles of the first order in the analytical regularization parameter arise (and cancel in the sum of two contributions) [7], we shall meet, for the non-planar diagram, poles up to the second order which are present in five contributions. These poles are also cancelled in the sum and we obtain a result which exists within dimensional regularization.

2. The Feynman integral for Fig. 1a can be written as

$$F_1(Q, m, \epsilon) = \int \int \frac{d^d k d^d l}{(l^2 - 2p_1 l + m^2)(l^2 - 2p_2 l + m^2)} \times \frac{1}{(k^2 - 2p_1 k + m^2)(k^2 - 2p_2 k + m^2)k^2(k-l)^2}. \quad (1)$$

We use dimensional regularization [12] with $d = 4 - 2\epsilon$. When presenting our results we shall omit $i\pi^{d/2}$ per loop and, when writing down separate contributions through expansion in ϵ , we shall also omit $\exp(-\gamma_E \epsilon)$ per loop (γ_E is the Euler constant).

Let us choose, for convenience, the external momenta as follows:

$$p_1 = \tilde{p}_1 + \frac{m^2}{Q^2} \tilde{p}_2, \quad p_2 = \tilde{p}_2 + \frac{m^2}{Q^2} \tilde{p}_1, \quad \tilde{p}_1 = (Q/2, -Q/2, 0, 0), \quad \tilde{p}_2 = (Q/2, Q/2, 0, 0) \quad (2)$$

so that $p_i^2 = m^2$, $\tilde{p}_i^2 = 0$, $2\tilde{p}_1 \tilde{p}_2 = 2\tilde{p}_1 p_2 = Q^2$. In the given limit, the following regions happen to be typical [9]:

$$\begin{aligned} \text{hard (h):} & \quad k \sim Q, \\ \text{1-collinear (1c):} & \quad k_+ \sim Q, \quad k_- \sim m^2/Q, \quad \underline{k} \sim m, \\ \text{2-collinear (2c):} & \quad k_- \sim Q, \quad k_+ \sim m^2/Q, \quad \underline{k} \sim m, \\ \text{ultrasoft (us):} & \quad k \sim m^2/Q. \end{aligned}$$

Here $k_{\pm} = k_0 \pm k_1$, $\underline{k} = (k_2, k_3)$. We mean by $k \sim Q$, etc. that any component of k_{μ} is of order Q .

One should consider any loop momentum k, l, \dots to be of one of the above types and allow for various choices of the loop momenta. (Still it is necessary to avoid double counting.) Other types of regions give zero contributions, in particular, when one of the loop momenta is *soft*, i.e. $k \sim m$. However, if some masses of the diagram were non-zero then some soft regions would generate non-zero contributions (that would start from a subleading order).

In the leading order, $1/Q^4$, we obtain contributions from the following nine regions: (h-h), (1c-h), (2c-h), (1c-1c), (2c-2c), (us-h), (us-1c), (us-2c), (us-us). In this list, regions for the loop momenta k and l in (1) are indicated in the first and the second place, respectively.

The (h-h) region generates terms obtained by Taylor expanding the integrand in the expansion parameter, m . In the leading order, this is nothing but the value of the massless planar diagram at $p_1^2 = p_2^2 = 0$ first evaluated in ref. [13]. Although the result can be expressed in gamma functions for general ϵ with the help of the method of integration by parts [14] (this was first done in [15]), we present it here, for brevity, in expansion in ϵ

$$C_{(h-h)}^{(1)} = \int \int \frac{d^d k d^d l}{(l^2 - 2\tilde{p}_1 l)(l^2 - 2\tilde{p}_2 l)(k^2 - 2\tilde{p}_1 k)(k^2 - 2\tilde{p}_2 k)k^2(k-l)^2} \quad (3)$$

$$= \left(\frac{1}{4\epsilon^4} + \frac{5\pi^2}{24\epsilon^2} + \frac{29\zeta(3)}{6\epsilon} + \frac{3\pi^4}{32} \right) \frac{1}{(Q^2)^{2+2\epsilon}}. \quad (4)$$

All contributions connected with the ultrasoft regions are easily evaluated in gamma functions by use of alpha parameters. In the leading order, we have

$$C_{(us-us)}^{(1)} = \int \int \frac{d^d k d^d l}{(-2\tilde{p}_1 l + m^2)(-2\tilde{p}_2 l + m^2)(-2\tilde{p}_1 k + m^2)(-2\tilde{p}_2 k + m^2)k^2(k-l)^2} = \frac{\Gamma(1-\epsilon)^2 \Gamma(2\epsilon)^2}{\epsilon^2 (-m^2)^{4\epsilon} (Q^2)^{2-2\epsilon}}, \quad (5)$$

$$C_{(us-h)}^{(1)} = \int \int \frac{d^d k d^d l}{(l^2 - 2\tilde{p}_1 l)(l^2 - 2\tilde{p}_2 l)(-2\tilde{p}_1 k + m^2)(-2\tilde{p}_2 k + m^2)k^2 l^2} = \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)\Gamma(\epsilon)^2\Gamma(-\epsilon)^2}{\Gamma(1-2\epsilon)(-m^2)^{2\epsilon}(Q^2)^2}, \quad (6)$$

$$C_{(us-1c)}^{(1)} = \int \int \frac{d^d k d^d l}{(-2\tilde{p}_1 l)(l^2 - 2p_2 l + m^2)(-2\tilde{p}_1 k + m^2)(-2\tilde{p}_2 k + m^2)} \times \frac{1}{k^2(l^2 - (2\tilde{p}_1 l)(2\tilde{p}_2 k)/Q^2)} = \frac{\Gamma(1-\epsilon)^2 \Gamma(\epsilon) \Gamma(2\epsilon) \Gamma(-\epsilon)}{\epsilon \Gamma(1-2\epsilon) (-m^2)^{3\epsilon} (Q^2)^{2-\epsilon}} \equiv C_{(us-2c)}^{(1)}. \quad (7)$$

Using alpha parameters, the rest contributions can be presented, for general ϵ , through Mellin-Barnes integrals

$$C_{(1c-1c)}^{(1)} = \int \int \frac{d^d k d^d l}{(-2\tilde{p}_1 l)(l^2 - 2p_2 l + m^2)(-2\tilde{p}_1 k)(k^2 - 2p_2 k + m^2)k^2(k-l)^2} = \frac{\Gamma(\epsilon)\Gamma(-\epsilon)\Gamma(2\epsilon)}{\Gamma(1+\epsilon)(-m^2)^{2\epsilon}(Q^2)^2}$$

$$\begin{aligned}
& \times \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(s-3\epsilon)\Gamma(s+1-2\epsilon)\Gamma(s+1-\epsilon)\Gamma(\epsilon-s)\Gamma(-s)}{\Gamma(s+1-3\epsilon)} \\
& \equiv C_{(2c-2c)}^{(1)}, \quad (8) \\
C_{(1c-h)}^{(1)} &= \int \int \frac{d^d k d^d l}{(l^2 - 2\tilde{p}_1 l)(l^2 - 2\tilde{p}_2 l)(-2\tilde{p}_1 k)(k^2 - 2p_2 k + m^2)} \\
& \times \frac{1}{k^2(l^2 - (2\tilde{p}_1 k)(2\tilde{p}_2 l)/Q^2)} = \frac{\Gamma(\epsilon)\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)(-m^2)^\epsilon(Q^2)^{2+\epsilon}} \\
& \times \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(s+1)\Gamma(s-\epsilon)\Gamma(s+1+\epsilon)\Gamma(-\epsilon-s)\Gamma(-s)}{\Gamma(s+1-2\epsilon)} \\
& \equiv C_{(2c-h)}^{(1)}. \quad (9)
\end{aligned}$$

We imply the standard way of choosing contours: the UV poles are to the right and the IR poles to the left of them. The above Mellin-Barnes integrals are expanded in ϵ by shifting the contours and picking up residua at points where UV and IR poles glue together when $\epsilon \rightarrow 0$. As a result we obtain

$$\begin{aligned}
& (Q^2)^2 \left[C_{(1c-1c)}^{(1)} + C_{(2c-2c)}^{(1)} + C_{(1c-h)}^{(1)} + C_{(2c-h)}^{(1)} \right] \\
& = -\frac{1}{2\epsilon^4} + \left(L^2 - \frac{\pi^2}{2} \right) \frac{1}{2\epsilon^2} + \left(\frac{1}{2}L^3 - \frac{\pi^2}{6}L - \frac{17\zeta(3)}{3} \right) \frac{1}{\epsilon} + \frac{7}{24}L^4 - 4\zeta(3)L - \frac{\pi^4}{144}. \quad (10)
\end{aligned}$$

where $L = \ln(Q^2/\mu^2)$ and we have put $\mu = 1$, for brevity. (Note that, in individual contributions, one has both $\ln(Q^2/\mu^2)$ and $\ln(\mu^2)$.)

Collecting all nine contributions together we observe that the poles in ϵ which turn out to be of very different (UV, IR and collinear) nature cancel, with the following result

$$(Q^2)^2 F_1(Q, m, 0) \stackrel{Q \rightarrow \infty}{\sim} \frac{1}{4}L^4 + \frac{\pi^2}{2}L^2 + \frac{7\pi^4}{60}, \quad (11)$$

in agreement with the leading order expansion of the well-known explicit result [11].

3. The expansion of the planar diagram Fig. 1a, with $m_1 = \dots = m_4 = 0$, $m_5 = m_6 = m$, in Limit 2 was obtained in arbitrary order, following the strategy of subtraction operators, in [7]. Note that the same expressions for all contributions of the expansion can be obtained with the help of the strategy of regions. The list of non-zero contributions, consists, in this language, of (h-h), (1c-h), (2c-h), (1c-1c) and (2c-2c) contributions plus a contribution that starts from the next-to-leading order and comes from the region where the momentum of the middle line is soft and the second loop momentum is considered to be hard.

Let us now consider the expansion of the non-planar diagram, Fig. 1b, in Limit 2. The Feynman integral can be written as

$$\begin{aligned}
F_2(Q, m, \epsilon) &= \int \int \frac{d^d k d^d l}{((k+l)^2 - 2p_1(k+l))((k+l)^2 - 2p_2(k+l))} \\
& \times \frac{1}{(k^2 - 2p_1 k)(l^2 - 2p_2 l)(k^2 - m^2)(l^2 - m^2)}, \quad (12)
\end{aligned}$$

where p_1 and p_2 satisfy the relations for $\tilde{p}_{1,2}$ in the previous section. We shall use as well the second choice of the loop momenta when k and l are chosen as momenta of lines 3 and 4, respectively, which is obtained by permutation of the masses and corresponds to (12) with $m_1 = m_2 = m_5 = m_6 = 0$, $m_3 = m_4 = m$.

Non-zero contributions to the expansion in the leading order are generated by the following regions: (h-h), (h-2c), (2c-h), (1c-1c), (2c-2c), (2c-1c), (1c-1c)', (2c-2c)' and (us-us)'. As above, we indicate the region for the loop momentum k in the first place and for l in the second place. We denote the regions for the second natural choice of the loop momenta by prime. The (h-h) contribution is given by the massless non-planar diagram. The result, in expansion in ϵ , can be found in [13]:

$$C_{(h-h)}^{(2)} = \left(\frac{1}{\epsilon^4} - \frac{\pi^2}{\epsilon^2} - \frac{83\zeta(3)}{3\epsilon} - \frac{59\pi^4}{120} \right) \frac{1}{(Q^2)^{2+2\epsilon}}. \quad (13)$$

The (us-us)' contribution is easily evaluated in gamma functions:

$$\begin{aligned} C_{(us-us)'}^{(2)} &= \int \int \frac{d^d k d^d l}{(-2p_1(k+l))(-2p_2(k+l))(-2p_1 k + m^2)(-2p_2 l + m^2)k^2 l^2} \\ &= \frac{1}{(Q^2)^{2-2\epsilon}(m^2)^{4\epsilon}} [\Gamma(\epsilon)\Gamma(2\epsilon)\Gamma(1-2\epsilon)]^2. \end{aligned} \quad (14)$$

The (2c-h) contribution is given by

$$\begin{aligned} C_{(2c-h)}^{(2)} &= \int \int \frac{d^d k d^d l}{(l^2 - 2p_1 l + (2\tilde{p}_2 k)(2\tilde{p}_1 l)/Q^2)(l^2 - 2p_2(k+l) + (2\tilde{p}_2 k)(2\tilde{p}_1 l)/Q^2)} \\ &\quad \times \frac{1}{(k^2 - 2p_1 k)(l^2 - 2p_2 l)(k^2 - m^2)l^2}, \end{aligned} \quad (15)$$

and the same leading order (h-2c) contribution is obtained by permutation of k and l . Using alpha parameters and (twice) Mellin-Barnes representation we obtain

$$C_{(h-2c)}^{(2)} = C_{(2c-h)}^{(2)} = \left(-\frac{3}{\epsilon^4} + \frac{\pi^2}{\epsilon^2} + \frac{22\zeta(3)}{\epsilon} + \frac{16\pi^4}{45} \right) \frac{1}{(Q^2)^{2+\epsilon}(m^2)^\epsilon}. \quad (16)$$

The (1c-1c) contribution is given by

$$\begin{aligned} C_{(1c-1c)}^{(2)} &= \int \int \frac{d^d k d^d l}{(-2p_1(k+l))((k+l)^2 - 2p_2(k+l))} \\ &\quad \times \frac{1}{(-2p_1 k)(l^2 - 2p_2 l)(k^2 - m^2)(l^2 - m^2)}, \end{aligned} \quad (17)$$

and the (2c-2c) contribution is obtained by permutation of k and l . We should also consider similar (1c-1c)' and (2c-2c)' contributions with the second choice of the loop momenta. The corresponding expressions are obtained by permutating the

masses (see above). The fifth non-zero contribution of the collinear-collinear type originates from the (2c-1c) region. It happens that these contributions are regulated dimensionally only in the sum. It is convenient to introduce an auxiliary analytic regularization into lines 3 and 4 by $\frac{1}{(k^2-2p_1k)^{1+x_1}(l^2-2p_2l)^{1+x_2}}$. In contrast to the planar two-loop diagram in this limit [5], we meet, in this example, poles in x_i up to the second order. In particular, the (2c-1c) contribution is evaluated in gamma functions, for general ϵ :

$$\begin{aligned} C_{(2c-1c)}^{(1)} &= \int \int \frac{d^d k d^d l}{(-2p_1l + (2p_2k)(2p_1l)/Q^2)(-2p_2k + (2p_2k)(2p_1l)/Q^2)} \\ &\quad \times \frac{1}{(k^2 - 2p_1k)(l^2 - 2p_2l)(k^2 - m^2)(l^2 - m^2)}, \\ &= \frac{\Gamma(x_1)\Gamma(x_2)\Gamma(-x_1 - \epsilon)\Gamma(-x_2 - \epsilon)\Gamma(x_1 + \epsilon)\Gamma(x_2 + \epsilon)}{\Gamma(1 + x_1)\Gamma(1 + x_2)\Gamma(-\epsilon)^2(-m^2)^{x_1+x_2+2\epsilon}(Q^2)^2}, \end{aligned} \quad (18)$$

Using the technique of alpha parameters and Mellin-Barnes representation for other four (e-e) contributions, we obtain, for each of them, a result in expansion in x_i . Then we switch off the analytic regularization (first, $x_2 \rightarrow x_1$ and then $x_1 \rightarrow 0$), observe that, in the sum of all the five contributions, the singular dependence in x_i drops out and obtain the following result in expansion in ϵ :

$$\begin{aligned} &(Q^2)^2 \left[C_{(1c-1c)}^{(2)} + C_{(2c-2c)}^{(2)} + C_{(1c-1c)'}^{(2)} + C_{(2c-2c)'}^{(2)} + C_{(2c-1c)}^{(2)} \right] \\ &= \frac{19}{4\epsilon^4} - \frac{9}{2\epsilon^3}L + \left(L^2 - \frac{11\pi^2}{4} \right) \frac{1}{2\epsilon^2} - \left(\frac{3\pi^2}{4}L + \frac{97\zeta(3)}{6} \right) \frac{1}{\epsilon} \\ &\quad + \frac{\pi^2}{12}L^2 + 9\zeta(3)L - \frac{23\pi^4}{32}, \end{aligned} \quad (19)$$

where $L = \ln(Q^2/m^2)$ and we have put $m = 1$, for brevity.

Collecting all the leading order contributions we see that the poles in ϵ are canceled and we arrive at the following result:

$$(Q^2)^2 F_2(Q, m, 0) \stackrel{Q \rightarrow \infty}{\sim} \frac{7}{12}L^4 - \frac{\pi^2}{2}L^2 + 20\zeta(3)L - \frac{31\pi^4}{180}. \quad (20)$$

On the expense of a computer algebra, it is possible to extend this result to any order in $1/Q^2$.

Acknowledgments. V.S. is grateful to M. Beneke, K.G. Chetyrkin and A.I. Davydchev for useful discussions.

References

- [1] S.G. Gorishny, preprints JINR E2-86-176, E2-86-177 (Dubna 1986); *Nucl. Phys.* B319 (1989) 633

- [2] K.G. Chetyrkin, *Teor. Mat. Fiz.* 75 (1988) 26; 76 (1988) 207; K.G. Chetyrkin, preprint MPI-PAE/PTh 13/91 (Munich, 1991)
- [3] V.A. Smirnov, *Commun. Math. Phys.* 134 (1990) 109; V.A. Smirnov, *Renormalization and asymptotic expansions* (Birkhäuser, Basel, 1991)
- [4] V.A. Smirnov, *Mod. Phys. Lett. A* 10 (1995) 1485; Talk given at the 5th International Conference on Physics beyond the Standard Model, Balholm, Norway (1997) [hep-ph/9708423]
- [5] V.A. Smirnov, *Phys. Lett.* B394 (1997) 205
- [6] A. Czarnecki and V.A. Smirnov, *Phys. Lett.* B394 (1997) 211
- [7] V.A. Smirnov, *Phys. Lett.* B404 (1997) 101
- [8] M. Beneke and V.A. Smirnov, *Nucl. Phys.* B522 (1998) 321
- [9] J.C. Collins, in *Perturbative QCD*, ed. A.H. Mueller, 1989, p. 573
- [10] V.V. Sudakov, *Zh. Eksp. Teor. Fiz.* 30 (1956) 87
- [11] N.I. Ussyukina and A.I. Davydychev, *Phys. Lett.* B298 (1993) 363
- [12] G. 't Hooft and M. Veltman, *Nucl. Phys.* B44 (1972) 189; C.G. Bollini and J.J. Giambiagi, *Nuovo Cim.* 12B (1972) 20
- [13] R.J. Gonsalves, *Phys. Rev. D* 28 (1983) 1542
- [14] K.G. Chetyrkin and F.V. Tkachov, *Nucl. Phys.* B192 (1981) 159
- [15] G. Kramer and B. Lampe, *J. Math. Phys.* 28 (1987) 945

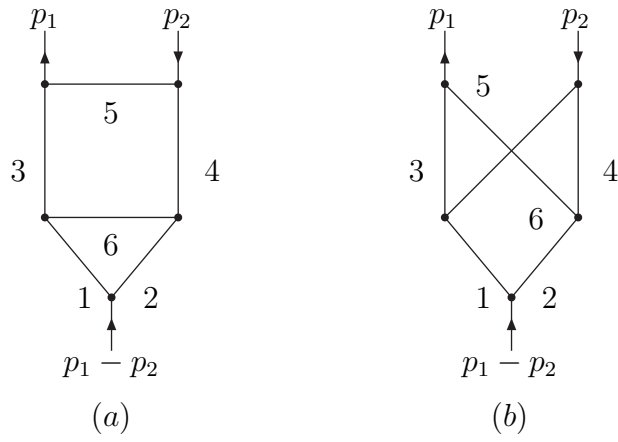


Figure 1: (a) Two-loop planar vertex diagram. (b) Two-loop non-planar vertex diagram.